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Dissipative quantum dynamics: solution of the generalized von Neumann equation for the damped harmonic oscillator

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Abstract. A recently proposed generalized von Neumann equation describing dissipative time evolution of quantum systems is applied to a damped and driven harmonic oscillator. Utilizing Lie algebraic methods the nonlinear operator equation is solved and from this solution the behaviour of the mean value $\langle\hat{x}\rangle$ is extracted, including both amplitude and phaseshift relative to the driving force. A differential equation for $\langle\hat{x}\rangle$ is derived and the damping constant is identified.

1. Quantum damping

In a preceding article [1], in the following denoted as paper I, the problem of dissipative quantum evolution was studied. A general discussion of the problems encountered in previous formulations of quantum dissipation can be found in paper I, where an alternative description by a generalized nonlinear von Neumann equation for the time evolution of the density operator has been proposed (equations closely related to the present ones have been advocated by Beretta [2–7]). In paper I, simple two- and three-state systems have been used to illustrate basic properties of the evolution equations. In the present article we will present an application of the proposed nonlinear equation to one of the celebrated physical model systems, namely the damped harmonic oscillator.

The quantum mechanical description of a damped harmonic oscillator has attracted a considerable amount of interest, and a critical comparison of the different approaches and results is far beyond the scope of the present article. A selection of relevant results can be found in [8–32].

A considerable amount of work has been devoted to the so-called Caldirola–Kanai Hamiltonian [33, 34]

$$H = -\frac{1}{2m}\,e^{-\gamma t}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}m\omega_0^2\,e^{\gamma t}\,x^2 \tag{1}$$

with the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left\{-\frac{1}{2m}\,e^{-\gamma t}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}m\omega_0^2\,e^{\gamma t}\,x^2\right\}\psi. \tag{2}$$

The Hamiltonian (1) can be motivated by a quantization of the classical equation of motion for the damped harmonic oscillator

$$\frac{\partial^2 x}{\partial t^2} + \gamma\frac{\partial x}{\partial t} + \omega_0^2 x = 0 \tag{3}$$

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with the Lagrange function

$$L=\frac{m}{2}\,e^{\gamma t}\,(\dot{x}^2-\omega_0^2x^2). \tag{4}$$

The problem remains that this derivation is not unique, due to the non-uniqueness of the Lagrangian (4). It is obvious that (1) describes simply the quantization of a classical particle with exponentially increasing mass and a dissipative system not at all. For a discussion of this topic see, for example, [8, 11, 16, 19, 21].

In the present paper the phenomenological nonlinear von Neumann equation for irreversible time evolution proposed in paper I is applied to the paradigmatic case of a damped harmonic oscillator exploring both the properties of the evolution equations and of the damped harmonic quantum oscillator. Section 2 recapitulates the basic theory of the nonlinear equations; in section 3 these equations are solved for the harmonic oscillator under free and constrained evolution. A few numerical studies illustrate properties of the obtained solutions. The paper concludes with a short resumé and an outlook.

2. The generalized von Neumann equation

An important property of irreversible processes is the increase of entropy during these processes. Nevertheless, the quantum mechanical evolution equation for the statistical operator $\hat{\rho}$, the von Neumann equation

$$\frac{d}{dt}\hat{\rho}=\frac{1}{i\hbar}[\hat{H},\hat{\rho}] \tag{5}$$

with the Hamiltonian $\hat{H}$ keeps the (von Neumann) entropy

$$\langle\hat{S}\rangle=-\mathrm{Tr}\,\hat{\rho}\ln\hat{\rho} \tag{6}$$

invariant. Therefore, in the preceding paper I the following generalization of (5) was proposed:

$$\frac{d}{dt}\hat{\rho}=\frac{1}{i\hbar}[\hat{H},\hat{\rho}]+\gamma\hat{D}(\hat{\rho}) \tag{7}$$

with $\gamma\in R_+$. $\hat{D}$ is the (super-) operator describing the dissipative part of time evolution. The requirements

$$\mathrm{Tr}\,\hat{\rho}=1 \tag{8}$$

$$\hat{\rho}^\dagger=\hat{\rho} \tag{9}$$

$$\hat{D}(\hat{\rho}_1\otimes\hat{\rho}_2)=\hat{D}(\hat{\rho}_1)\otimes\hat{\rho}_2+\hat{\rho}_1\otimes\hat{D}(\hat{\rho}_2) \tag{10}$$

(the last equation follows from

$$\frac{d}{dt}(\hat{\rho}_1\otimes\hat{\rho}_2)=\left(\frac{d}{dt}\hat{\rho}_1\right)\otimes\hat{\rho}_2+\hat{\rho}_1\otimes\left(\frac{d}{dt}\hat{\rho}_2\right)$$

for independent, uncorrelated systems 1, 2) are not sufficient to determine $\hat{D}$ uniquely. There are different forms of $\hat{D}$ satisfying the above equations, for example

$$\hat{D}(\hat{\rho})=(\hat{S}-\langle\hat{S}\rangle)\hat{\rho} \tag{11}$$

with the entropy operator

$$\hat{S} = -\ln \hat{\rho} \tag{12}$$

and

$$\hat{D}_{\hat{A}}(\hat{\rho}) = \tfrac{1}{2}[\hat{A}, \hat{\rho}]_+ - \langle \hat{A} \rangle \hat{\rho} \tag{13}$$

with an arbitrary self-adjoint operator $\hat{A}$($[\,,]_+$ is the anticommutator). Both forms as well as linear combinations will be used in the following.

For free dissipative time evolution

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \frac{1}{\mathrm{i}\hbar}[\hat{H}, \hat{\rho}] - \gamma(\hat{S} - \langle \hat{S} \rangle)\hat{\rho} \tag{14}$$

the von Neumann entropy is non-decreasing,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{S} \rangle = \mathrm{Tr}\left(\hat{S}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}\right) = \gamma(\langle \hat{S}^2 \rangle - \langle \hat{S} \rangle^2) \geqslant 0 \tag{15}$$

because γ is positive.

Using the parameter differentiation technique developed by Wilcox [35], (14) can be converted into the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{S} = \frac{1}{\mathrm{i}\hbar}[\hat{H}, \hat{S}] - \gamma(\hat{S} - \langle \hat{S} \rangle) \tag{16}$$

for the entropy operator $\hat{S}$ [1]. One immediately deduces that the equipartition $\hat{S} = \langle \hat{S} \rangle \hat{1}$ is stationary solution of (16) (if we consider finite-dimensional Hilbert spaces, where the normalization is guaranteed). Equation (16) will be solved for the harmonic oscillator in the next section.

For a system in contact with a heat bath of inverse temperature $\beta = 1/kT$, a generalization of (14) has been suggested in paper I, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \frac{1}{\mathrm{i}\hbar}[\hat{H}, \hat{\rho}] + \gamma[(\hat{S} - \langle \hat{S} \rangle)\hat{\rho} - \beta(\tfrac{1}{2}[\hat{H}, \hat{\rho}]_+ - \langle \hat{H} \rangle \hat{\rho})] \tag{17}$$

where the canonical distribution

$$\hat{\rho}_c = \frac{\mathrm{e}^{-\beta \hat{H}}}{Z} \tag{18}$$

is a stationary solution.

According to (17) the mean value of an observable $\hat{B}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{B} \rangle = \frac{1}{\mathrm{i}\hbar}\langle [\hat{B}, \hat{H}] \rangle + \gamma[\langle \hat{B}\hat{S} \rangle - \langle \hat{B} \rangle \langle \hat{S} \rangle - \beta(\tfrac{1}{2}\langle [\hat{B}, \hat{H}]_+ \rangle - \langle \hat{B} \rangle \langle \hat{H} \rangle)] + \left\langle \frac{\partial \hat{B}}{\partial t} \right\rangle. \tag{19}$$

For the Hamiltonian $\hat{H}$ and the entropy $\hat{S}$ we obtain in particular

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{H} \rangle = \gamma[\langle \hat{H}\hat{S} \rangle - \langle \hat{H} \rangle \langle \hat{S} \rangle - \beta(\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2)] + \left\langle \frac{\partial \hat{H}}{\partial t} \right\rangle \tag{20}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{S} \rangle = \gamma[\langle \hat{S}^2 \rangle - \langle \hat{S} \rangle^2 - \beta(\langle \hat{H}\hat{S} \rangle - \langle \hat{H} \rangle \langle \hat{S} \rangle)]. \tag{21}$$

Equation (21) shows that the entropy is not generally an increasing function of time for the evolution (17).

3. The damped harmonic oscillator

In this section we discuss the application of (16) and (17) to the forced harmonic oscillator

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2}) - F(t)\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger+\hat{a}) \tag{22}$$

with creation and annihilation operators

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} - \mathrm{i}\frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}\right) \tag{23}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + \mathrm{i}\frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}\right). \tag{24}$$

$F(t)$ is the driving force, for which we assume in the following the form

$$F(t) = F_0\cos\Omega t. \tag{25}$$

3.1. Free dissipative evolution

To solve (16) we will use the concept of a dynamical Lie algebra ([36, 37] and references therein). This dynamical algebra is defined as the smallest Lie algebra $\mathscr{L}$ generated by the Hamiltonian of the system. With abbreviations $\hat{\Gamma}_0=\hat{1}$, $\hat{\Gamma}_1=\hat{a}^\dagger\hat{a}+\frac{1}{2}$, $\hat{\Gamma}_2=\hat{a}^\dagger$ and $\hat{\Gamma}_3=\hat{a}$, and from the well known commutator relations

$$[\hat{\Gamma}_1,\hat{\Gamma}_2] = [\hat{a}^\dagger\hat{a}+\tfrac{1}{2},\hat{a}^\dagger] = \hat{a}^\dagger = \hat{\Gamma}_2 \tag{26}$$

$$[\hat{\Gamma}_1,\hat{\Gamma}_3] = [\hat{a}^\dagger\hat{a}+\tfrac{1}{2},\hat{a}] = -\hat{a} = -\hat{\Gamma}_3 \tag{27}$$

$$[\hat{\Gamma}_2,\hat{\Gamma}_3] = [\hat{a}^\dagger,\hat{a}] = \hat{1} = \hat{\Gamma}_0 \tag{28}$$

we immediately deduce

$$\mathscr{L} = \{\hat{1},\hat{a}^\dagger\hat{a}+\tfrac{1}{2},\hat{a}^\dagger,\hat{a}\} = \{\hat{\Gamma}_0,\hat{\Gamma}_1,\hat{\Gamma}_2,\hat{\Gamma}_3\}. \tag{29}$$

The non-vanishing structure constants C^k_{ij}, defined by $[\hat{\Gamma}_i,\hat{\Gamma}_j]=\Sigma_k C^k_{ij}\hat{\Gamma}_k$, are $C^0_{32}=-C^0_{23}=C^2_{12}=-C^2_{21}=C^3_{31}=-C^3_{31}=-C^3_{13}=1$. In this notation the Hamiltonian (22) is written as

$$\hat{H} = \sum_{i=1}^{3} h_i\hat{\Gamma}_i \tag{30}$$

with $\hbar\omega=h_1$, $h_2(t)=h_3(t)=h(t)=-F(t)\sqrt{\hbar/2m\omega}$. Assuming that the entropy operator $\hat{S}$ is an element of the dynamical algebra

$$\hat{S} = \sum_{k=0}^{3}\lambda_k\hat{\Gamma}_k \tag{31}$$

(16) can be easily solved:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{S} &= \sum_{k=0}^{3}\dot{\lambda}_k\hat{\Gamma}_k = \frac{1}{\mathrm{i}\hbar}[\hat{H},\hat{S}] - \gamma(\hat{S}-\langle\hat{S}\rangle\hat{\Gamma}_0)\\ &= \frac{1}{\mathrm{i}\hbar}\sum_{i,j,k=1}^{3}h_i\lambda_jC^k_{ij}\hat{\Gamma}_k - \gamma\left(\sum_{k=1}^{3}\lambda_k\hat{\Gamma}_k - \sum_{k=1}^{3}\lambda_k\langle\hat{\Gamma}_k\rangle\hat{\Gamma}_0\right).\end{aligned} \tag{32}$$

Comparing coefficients we obtain

$$\dot{\lambda}_0=\frac{1}{\mathrm{i}\hbar}h(t)(\lambda_2-\lambda_3)+\gamma\sum_{j=1}^{3}\lambda_j\langle\hat{\Gamma}_j\rangle \tag{33}$$

$$\dot{\lambda}_1=-\gamma\lambda_1 \tag{34}$$

$$\dot{\lambda}_2=-(\gamma+\mathrm{i}\omega)\lambda_2+\frac{\mathrm{i}}{\hbar}h(t)\lambda_1 \tag{35}$$

$$\dot{\lambda}_3=-(\gamma-\mathrm{i}\omega)\lambda_3-\frac{\mathrm{i}}{\hbar}h(t)\lambda_1. \tag{36}$$

Since

$$\lambda_0=\ln\mathrm{Tr}\left[\exp\left(-\sum_{k=1}^{3}\lambda_k\hat{\Gamma}_k\right)\right] \tag{37}$$

is uniquely determined by the normalization of $\hat{\rho}$, we can omit (33) in the following. Self-adjointness of $\hat{S}$ requires

$$\bar{\lambda}_1=\lambda_1 \tag{38}$$

$$\bar{\lambda}_2=\lambda_3 \tag{39}$$

which is conserved in time since $\dot{\lambda}_2=\dot{\bar{\lambda}}_3$ and $\dot{\lambda}_1=\dot{\bar{\lambda}}_1$ from (34)–(36).

From (35) and (36) we easily obtain differential equations for the real and imaginary parts $\mathrm{Re}\,\lambda_2$ and $\mathrm{Im}\,\lambda_2$:

$$\mathrm{Re}\,\dot{\lambda}_2=-\gamma\,\mathrm{Re}\,\lambda_2+\omega\,\mathrm{Im}\,\lambda_2 \tag{40}$$

$$\mathrm{Im}\,\dot{\lambda}_2=-\gamma\,\mathrm{Im}\,\lambda_2-\omega\,\mathrm{Re}\,\lambda_2+\frac{1}{\hbar}h(t)\lambda_1. \tag{41}$$

Equations (34)–(36) have the solutions

$$\lambda_1(t)=C_1\,\mathrm{e}^{-\gamma t}=\lambda_1(0)\,\mathrm{e}^{-\gamma t} \tag{42}$$

$$\lambda_2(t)=\left(C_2+C_1\frac{\mathrm{i}}{\hbar}\int_0^t h(t)\,\mathrm{e}^{\mathrm{i}\omega t'}\,\mathrm{d}t'\right)\mathrm{e}^{-(\gamma+\mathrm{i}\omega)t} \tag{43}$$

$$\lambda_3(t)=\left(C_3-C_1\frac{\mathrm{i}}{\hbar}\int_0^t h(t)\,\mathrm{e}^{-\mathrm{i}\omega t'}\,\mathrm{d}t'\right)\mathrm{e}^{-(\gamma-\mathrm{i}\omega)t}. \tag{44}$$

The constants C_i are determined by the initial conditions ($\bar{C}_2=C_3$).

Because of the exponential factor $\mathrm{e}^{-\gamma t}$, (42)–(44) predict a convergence of $\hat{\rho}$ to the equipartition distribution $\hat{1}/\mathrm{Tr}\,i$. But this expression is only defined for finite-dimensional Hilbert spaces since otherwise $\mathrm{Tr}\,\hat{1}$ is divergent, and convergence to equipartition alone is not very meaningful. It is more instructive to consider the mean values of physical observables.

For the case of the position operator $\hat{x}$ we have to consider the time evolution of

$$\langle\hat{x}\rangle=\mathrm{Tr}\,\hat{\rho}\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}\,\mathrm{Tr}\,\hat{\rho}(\hat{a}^{\dagger}+\hat{a}). \tag{45}$$

The expectation values $\mathrm{Tr}\,\hat\rho\hat\Gamma_2$, $\mathrm{Tr}\,\hat\rho\hat\Gamma_3$ and, for completeness, $\mathrm{Tr}\,\hat\rho\hat\Gamma_1$ are derived in appendix 1 using the representation

$$\hat\rho = \mathrm{e}^{-\hat S} = \exp\left(-\sum_{i=0}^{3} \lambda_i \hat\Gamma_i\right) \tag{46}$$

resulting in

$$\mathrm{Tr}\,\hat\rho\hat\Gamma_1 = \langle\hat\Gamma_1\rangle = \frac{1}{2} + \frac{1}{\mathrm{e}^{\lambda_1}-1} + \frac{\lambda_2\lambda_3}{\lambda_1^2} \tag{47}$$

$$\mathrm{Tr}\,\hat\rho\hat\Gamma_2 = \langle\hat\Gamma_2\rangle = -\frac{\lambda_3}{\lambda_1} \tag{48}$$

$$\mathrm{Tr}\,\hat\rho\hat\Gamma_3 = \langle\hat\Gamma_3\rangle = -\frac{\lambda_2}{\lambda_1}. \tag{49}$$

Using $\bar\lambda_2 = \lambda_3$ and $\bar\lambda_1 = \lambda_1$ we get

$$\langle\hat x\rangle = -\sqrt{\frac{\hbar}{2m\omega}}\,\frac{2\,\mathrm{Re}\,\lambda_2}{\lambda_1}. \tag{50}$$

From (50), (40) and (41) one easily obtains a differential equation for the mean value $\langle\hat x\rangle$. Differentiation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat x\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\sqrt{\frac{2\hbar}{m\omega}}\,\frac{\mathrm{Re}\,\lambda_2}{\lambda_1}\right) = \left(-\frac{\dot\lambda_2}{\lambda_1} + \frac{\mathrm{Re}\,\lambda_2\dot\lambda_1}{\lambda_1^2}\right)\sqrt{\frac{2\hbar}{m\omega}}\\ &= -\frac{\omega\,\mathrm{Im}\,\lambda_2}{\lambda_1}\sqrt{\frac{2\hbar}{m\omega}}\end{aligned} \tag{51}$$

and a second differentiation

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle\hat x\rangle &= \left(-\frac{\omega\,\mathrm{Im}\,\dot\lambda_2}{\lambda_1} + \frac{\omega\,\mathrm{Im}\,\lambda_2\dot\lambda_1}{\lambda_1^2}\right)\sqrt{\frac{2\hbar}{m\omega}}\\ &= -\omega^2\langle\hat x\rangle + \frac{F(t)}{m}\end{aligned} \tag{52}$$

shows that $\langle\hat x\rangle$ obeys the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle\hat x\rangle + \omega^2\langle\hat x\rangle = \frac{F(t)}{m} \tag{53}$$

which is independent of γ. This is exactly the equation of an undamped driven harmonic oscillator with driving force $F(t)$, which can also be derived from the non-dissipative von Neumann equation (5), i.e. for $\gamma = 0$. Furthermore, it agrees with the classical equation of motion because of Ehrenfest's theorem. Therefore, the term $\gamma(\hat S - \langle\hat S\rangle)\hat\rho$ in the generalized von Neumann equation does not influence the average displacement $\langle\hat x\rangle$. The average energy transfer $\langle H_0\rangle(t) = \langle\hbar\omega(\hat a^\dagger\hat a + \frac{1}{2})\rangle$ and the dispersion $\langle(\hat x - \langle x\rangle)^2\rangle$ are, however, γ dependent and diverge in the limit $t\to\infty$, which is also clear because $\hat\rho$ approaches the equipartition distribution in this limit.

For the harmonic driving force (25), (43) can be easily integrated in closed form:

$$\lambda_2 = \left(C_2\,\mathrm{e}^{-\mathrm{i}\omega t} - \frac{C_1F_0}{\Omega^2-\omega^2}\sqrt{\frac{1}{2m\omega\hbar}}\,[\omega\,(\cos\omega t - \cos\Omega t) + \mathrm{i}(\Omega\sin\Omega t - \omega\sin\omega t)]\right)\mathrm{e}^{-\gamma t} \tag{54, 55}$$

for $\Omega \neq \omega$ and

$$\lambda_2 = \left\{ C_2\,\mathrm{e}^{-\mathrm{i}\omega t} - C_1 F_0 \sqrt{\frac{1}{8m\omega\hbar}} \left[t \sin \omega t + \mathrm{i} \left(\frac{\sin \omega t}{\omega} + t \cos \omega t \right) \right] \right\} \mathrm{e}^{-\gamma t} \tag{56}$$

at resonance ($\Omega = \omega$). With initial value $\lambda_2(0) = 0$ (i.e. an oscillator initially at rest) we obtain

$$\operatorname{Re} \lambda_2(t) = -C_1 F_0 \sqrt{\frac{1}{2\hbar m\omega}}\, \omega \frac{\cos \omega t - \cos \Omega t}{\Omega^2 - \omega^2} \mathrm{e}^{-\gamma t} \tag{57}$$

for $\Omega \neq \omega$ and

$$\operatorname{Re} \lambda_2(t) = -C_1 F_0 \sqrt{\frac{1}{8\hbar m\omega}}\, t \sin \omega t\, \mathrm{e}^{-\gamma t} \tag{58}$$

for $\Omega = \omega$. The average displacement $\langle \hat{x} \rangle$ can therefore be written as

$$\langle \hat{x} \rangle = \frac{F_0}{m} \frac{\cos \omega t - \cos \Omega t}{\Omega^2 - \omega^2} = \frac{2F_0}{m} \frac{\sin[(\Omega + \omega t/2] \sin[(\Omega - \omega)t/2]}{\Omega^2 - \omega^2} \tag{59}$$

for $\Omega \neq \omega$ and

$$\langle \hat{x} \rangle = \frac{F_0}{2m} \frac{\sin \omega t}{\omega} t \tag{60}$$

on resonance.

3.2. *The damped evolution*

In this section the evolution equation (17),

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{\rho} = \frac{1}{\mathrm{i}\hbar} [\hat{H}, \hat{\rho}] + \gamma[(\hat{S} - \langle \hat{S} \rangle) \hat{\rho} - \beta(\tfrac{1}{2}[\hat{H}, \hat{\rho}]_+ - \langle \hat{H} \rangle \hat{\rho})] \tag{61}$$

will be solved. The anticommutator $[\hat{H}, \hat{\rho}]_+$ precludes a conversion into a differential equation for $\hat{S}$ as for the free dissipative evolution (16). Here we directly attack (61) using coherent states $|\alpha\rangle$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{62}$$

with an arbitrary complex number α. $|\alpha\rangle$ can be written as

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \mathrm{e}^{-|\alpha|^2/2}\, \mathrm{e}^{\alpha \hat{a}^\dagger} |0\rangle \tag{63}$$

in terms of eigenstates $|n\rangle$ of $\hat{a}^\dagger \hat{a} : \hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle$. From (63) one immediately finds

$$\langle \alpha | \hat{a}^\dagger = \bar{\alpha} \langle \alpha | \tag{64}$$

$$\langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = \alpha\bar{\alpha} = |\alpha|^2. \tag{65}$$

For $\hat{\rho}$ we will again use the exponential form

$$\hat{\rho} = \exp\left(-\sum_{i=0}^{3} \lambda_i \hat{\Gamma}_i \right) \tag{66}$$

as well as the alternative product of exponentials

$$\hat{\rho}=\prod_{i=0}^{3} e^{-g_i\hat{\Gamma}_i}=e^{-g_0\hat{\Gamma}_0}\,e^{-g_1\hat{\Gamma}_1}\,e^{-g_2\hat{\Gamma}_2}\,e^{-g_3\hat{\Gamma}_3}. \tag{67}$$

The two sets of coefficients are interrelated by (see appendix 1)

$$g_1=\lambda_1 \tag{68}$$

$$g_2=\frac{\lambda_2}{\lambda_1}(e^{\lambda_1}-1) \tag{69}$$

$$g_3=\frac{\lambda_3}{\lambda_1}(1-e^{-\lambda_1}) \tag{70}$$

$$g_0=\lambda_0+\frac{\lambda_2\lambda_3}{\lambda_1^2}(1-\lambda_1-e^{-\lambda_1}) \tag{71}$$

with the Hermiticity constraints

$$\bar{g}_1=g_1 \tag{72}$$

$$g_3=e^{-g_1}\,\bar{g}_2. \tag{73}$$

The inverse relations are

$$\lambda_1=g_1 \tag{74}$$

$$\lambda_2=\frac{g_2g_1}{e^{g_1}-1} \tag{75}$$

$$\lambda_3=\frac{g_3g_1}{1-e^{-g_1}}. \tag{76}$$

After these preliminary remarks we now begin solving (61). First we write down the left- and right-hand sides of (61) using the representations $\hat{S}=\Sigma_{i=0}^{3}\lambda_i\hat{\Gamma}_i$ and $\hat{\rho}=\Pi_{i=0}^{e}\,e^{-g_i\hat{\Gamma}_i})$. Then we evaluate matrix elements $\langle\alpha|\ldots|\alpha\rangle$, which—after some appropriate transformations—turn out be useful.

With the product representation (67) one gets

$$\begin{aligned}\frac{d}{dt}\hat{\rho}&=\frac{d}{dt}\left(\prod_{i=0}^{3} e^{-g_i\hat{\Gamma}_i}\right)\\&=-\dot{g}_0\hat{\Gamma}_0\hat{\rho}-\dot{g}_1\hat{\Gamma}_1\hat{\rho}-e^{-g_0\hat{\Gamma}_0}\,e^{-g_2\hat{\Gamma}_1}\,\dot{g}_2\hat{\Gamma}_2\,e^{-g_3\hat{\Gamma}_3}-\hat{\rho}\dot{g}_3\hat{\Gamma}_3.\end{aligned} \tag{77}$$

Matrix elements $\langle\alpha|\ldots|\alpha\rangle$ of (77) can be formed conveniently by normal ordering of (77). Here we prefer to convert all matrix elements into expressions proportional to $\langle\alpha|\hat{\rho}|\alpha\rangle$. Inserting the defintions of the $\hat{\Gamma}_i$ into (77) we get

$$\begin{aligned}\left\langle\alpha\left|\frac{d}{dt}\hat{\rho}\right|\alpha\right\rangle\\&=(-\dot{g}_0-\tfrac{1}{2}\dot{g}_1)\langle\alpha|\hat{\rho}|\alpha\rangle-\dot{g}_3\langle\alpha|e^{-g_0\hat{\Gamma}_0}\,e^{-g_1\hat{\Gamma}_1}\,e^{-g_2\hat{\Gamma}_2}\,e^{-g_3\hat{\Gamma}_3}\,\hat{a}|\alpha\rangle\\&\quad-\dot{g}_1\langle\alpha|\hat{a}^\dagger\hat{a}\,e^{-g_0\hat{\Gamma}_0}\,e^{-g_1\hat{\Gamma}_1}\,e^{-g_2\hat{\Gamma}_2}\,e^{-g_3\hat{\Gamma}_3}|\alpha\rangle\\&\quad-\dot{g}_2\langle\alpha|e^{-g_0\hat{\Gamma}_0}\,e^{-g_1\hat{\Gamma}_1}\,\hat{a}^\dagger\,e^{-g_2\hat{\Gamma}_2}\,e^{-g_3\hat{\Gamma}_3}|\alpha\rangle\\&=(-\dot{g}_0-\tfrac{1}{2}\dot{g}_1-\alpha\dot{g}_3)\langle\alpha|\hat{\rho}|\alpha\rangle-\dot{g}_1\bar{\alpha}\langle\alpha|\hat{a}\hat{\rho}|\alpha\rangle\\&\quad-\dot{g}_2\langle\alpha|e^{-g\hat{\Gamma}_0}\,e^{-g_1\hat{\Gamma}_1}\,\hat{a}^\dagger\,e^{-g_2\hat{\Gamma}_2}\,e^{-g_3\hat{\Gamma}_3}|\alpha\rangle\end{aligned} \tag{78}$$

and evaluating the right-hand side of (61) we obtain

$$\left\langle \alpha \left| \frac{\mathrm{d}}{\mathrm{d}t} \hat{\rho} \right| \alpha \right\rangle$$
$$= \langle \alpha | \hat{H}\hat{\rho} | \alpha \rangle \left(\frac{1}{\mathrm{i}\hbar} - \frac{\gamma\beta}{2} \right) + \langle \alpha | \hat{\rho}\hat{H} | \alpha \rangle \left(-\frac{1}{\mathrm{i}\hbar} - \frac{\gamma\beta}{2} \right)$$
$$+ \gamma \langle \alpha | \hat{S}\hat{\rho} | \alpha \rangle - \gamma(\langle \hat{S} \rangle - \beta \langle \hat{H} \rangle) \langle \alpha | \hat{\rho} | \alpha \rangle$$
$$= \left(\frac{1}{\mathrm{i}\hbar} - \frac{\gamma\beta}{2} \right) \langle \alpha | \hat{H}\hat{\rho} | \alpha \rangle - \left(\frac{1}{\mathrm{i}\hbar} + \frac{\gamma\beta}{2} \right) \langle \alpha | \hat{\rho}\hat{H} | \alpha \rangle$$
$$- \gamma((\langle \hat{S} \rangle - \beta \langle \hat{H} \rangle) \langle \alpha | \hat{\rho} | \alpha \rangle - \langle \alpha | \hat{S}\hat{\rho} | \alpha \rangle). \tag{79}$$

For $\langle \alpha | \hat{H}\hat{\rho} | \alpha \rangle$ and $\langle \alpha | \hat{S}\hat{\rho} | \alpha \rangle$ this yields

$$\langle \alpha | \hat{H}\hat{\rho} | \alpha \rangle = \sum_{i=1}^{3} h_i \langle \alpha | \hat{\Gamma}_i \hat{\rho} | \alpha \rangle = \left(\frac{\hbar\omega}{2} + h(t)\bar{\alpha} \right) \langle \alpha | \hat{\rho} | \alpha \rangle + (h(t) + \hbar\omega\bar{\alpha}) \langle \alpha | \hat{a}\hat{\rho} | \alpha \rangle \tag{80}$$

$$\langle \alpha | \hat{\rho}\hat{H} | \alpha \rangle = \sum_{i=1}^{3} h_i \langle \alpha | \hat{\rho}\hat{\Gamma}_i | \alpha \rangle = \left(\frac{\hbar\omega}{2} + h(t)\alpha \right) \langle \alpha | \hat{\rho} | \alpha \rangle + (h(t) + \hbar\omega\alpha) \langle \alpha | \hat{\rho}\hat{a}^\dagger | \alpha \rangle \tag{81}$$

$$\langle \alpha | \hat{S}\hat{\rho} | \alpha \rangle = \sum_{i=0}^{3} \lambda_i \langle \alpha | \hat{\Gamma}_i \hat{\rho} | \alpha \rangle = (\lambda_0 + \tfrac{1}{2}\lambda_1 + \bar{\alpha}\lambda_2) \langle \alpha | \hat{\rho} | \alpha \rangle + (\bar{\alpha}\lambda_1 + \lambda_3) \langle \alpha | \hat{a}\hat{\rho} | \alpha \rangle \tag{82}$$

where we have used the representations (30) and (31) for the Hamiltonian and the entropy operator. If we were able to convert $\hat{\rho}\hat{a}^\dagger$ into $\hat{a}^\dagger\hat{\rho}$, $\hat{a}\hat{\rho}$ into $\hat{\rho}\hat{a}$ and, finally, $\mathrm{e}^{-g_0\hat{\Gamma}_0}\mathrm{e}^{-g_1\hat{\Gamma}_1}\hat{a}^\dagger\mathrm{e}^{-g_2\hat{\Gamma}_2}\mathrm{e}^{-g_3\hat{\Gamma}_3}$ into $\hat{a}^\dagger\hat{\rho}$ we could omit the matrix element $\langle \alpha | \hat{\rho} | \alpha \rangle$ from (79) to get an expression which solely depends on the coefficients g_i, $\dot{g}_i$, h_i and λ_i as well as on α. Here the $\dot{g}_i$ appear only linear. From this equation we will deduce differential equations $\dot{g}_i = X_i(g_0, \ldots, g_3)$ for the coefficient functions g_i. It is shown in appendix 2 that the relevant matrix elements are

$$\langle \alpha | \hat{\rho}\hat{a}^\dagger | \alpha \rangle = (\bar{\alpha} - \overline{g_2})\,\mathrm{e}^{-g_1} \langle \alpha | \hat{\rho} | \alpha \rangle \tag{83}$$

$$\langle \alpha | \hat{a}\hat{\rho} | \alpha \rangle = (\alpha - g_2)\,\mathrm{e}^{-g_1} \langle \alpha | \hat{\rho} | \alpha \rangle \tag{84}$$

$$\langle \alpha | \mathrm{e}^{-g_0\hat{\Gamma}_0}\mathrm{e}^{-g_1\hat{\Gamma}_1}\hat{a}^\dagger\mathrm{e}^{-g_2\hat{\Gamma}_2}\mathrm{e}^{-g_3\hat{\Gamma}_3} | \alpha \rangle = \bar{\alpha}\,\mathrm{e}^{-g_1} \langle \alpha | \hat{\rho} | \alpha \rangle. \tag{85}$$

Because of $\langle \alpha | \hat{\rho} | \alpha \rangle \neq 0$ (see appendix 2) we obtain from (78)–(85)

$$-\dot{g}_0 - \tfrac{1}{2}\dot{g}_1 - \alpha\dot{g}_3 - \dot{g}_1\bar{\alpha}(\alpha - g_2)\,\mathrm{e}^{-g_1} - \dot{g}_2\bar{\alpha}\,\mathrm{e}^{-g_1}$$
$$= \left(\frac{1}{\mathrm{i}\hbar} - \frac{\gamma\beta}{2} \right) \left[\left(\frac{\hbar\omega}{2} + h(t)\bar{\alpha} \right) + (h(t) + \hbar\omega\bar{\alpha})(\alpha - g_2)\,\mathrm{e}^{-g_1} \right]$$
$$+ \left(-\frac{1}{\mathrm{i}\hbar} - \frac{\gamma\beta}{2} \right) \left[\left(\frac{\hbar\omega}{2} + h(t)\alpha \right) + (h(t) + \hbar\omega\alpha)(\bar{\alpha} - \overline{g_2})\,\mathrm{e}^{-g_1} \right]$$
$$- \gamma[(\langle \hat{S} \rangle - \beta \langle \hat{H} \rangle) - ((\lambda_0 + \tfrac{1}{2}\lambda_1 + \bar{\alpha}\lambda_2) + (\bar{\alpha}\lambda_1 + \lambda_3)(\alpha - g_2)\,\mathrm{e}^{-g_1})] \tag{86}$$

and, finally,

$$-\dot{g}_0 - \tfrac{1}{2}\dot{g}_1 + \dot{g}_1\,\mathrm{e}^{-g_1}[2\,\mathrm{Re}(\bar{\alpha}g_2) - \alpha\bar{\alpha}] - 2\,\mathrm{e}^{-g_1}\,\mathrm{Re}(\bar{\alpha}\dot{g}_2)$$
$$= \frac{2}{\hbar}\{h(t)\,\mathrm{Im}[\bar{\alpha}(1 - \mathrm{e}^{-g_1})] - \mathrm{e}^{-g_1}[h(t)\,\mathrm{Im}\,g_2 + \hbar\omega\,\mathrm{Im}(\bar{\alpha}g_2)]\}$$
$$- \gamma\beta\{\hbar\omega(\tfrac{1}{2} + \alpha\bar{\alpha}\,\mathrm{e}^{-g_1}) + h(t)(1 + e^{-g_1})\,\mathrm{Re}\,\alpha$$
$$- \mathrm{e}^{-g_1}[h(t)\,\mathrm{Re}\,g_2 + \hbar\omega\,\mathrm{Re}(\bar{\alpha}g_2)]\}$$
$$- \gamma[\langle \hat{S} \rangle - \beta \langle \hat{H} \rangle - \lambda_0 - \lambda_1(\tfrac{1}{2} + \alpha\bar{\alpha}\,\mathrm{e}^{-g_1})$$
$$- 2\,\mathrm{e}^{-g_1}\,\mathrm{Re}(\bar{\alpha}\lambda_2) + \lambda_3 g_2\,\mathrm{e}^{-g_1}]. \tag{87}$$

This equation is valid for all complex values of α. Inserting, for example, the five special α-values 0, ±1, $\pm$i into (87) yields five equations which must simultaneously be fulfilled:

$\alpha=0$,

$$-\dot{g}_0-\tfrac{1}{2}\dot{g}_1=-\frac{2}{\hbar}\,\mathrm{e}^{-g_1}\,h(t)\,\mathrm{Im}\,g_2-\gamma\beta\left(\frac{\hbar\omega}{2}-h(t)\,\mathrm{Re}\,g_2\,\mathrm{e}^{-g_1}\right)$$
$$-\gamma(\langle\hat{S}\rangle-\beta\langle\hat{H}\rangle-\lambda_0-\tfrac{1}{2}\lambda_1+\lambda_3 g_2\,\mathrm{e}^{-g_1}) \tag{88}$$

$\alpha=\pm1$,

$$-\dot{g}_0-\tfrac{1}{2}\dot{g}_1+\dot{g}_1\,\mathrm{e}^{-g_1}(\pm2\,\mathrm{Re}\,g_2-1)\mp2\,\mathrm{e}^{-g_1}\,\mathrm{Re}\,\dot{g}_2$$
$$=-\frac{2}{\hbar}\,\mathrm{e}^{-g_1}(h(t)\pm\hbar\omega)\,\mathrm{Im}\,g_2$$
$$-\gamma\beta[\hbar\omega(\tfrac{1}{2}+\mathrm{e}^{-g_1})\pm h(t)(1+\mathrm{e}^{-g_1})-\mathrm{e}^{-g_1}(h(t)\pm\hbar\omega)\,\mathrm{Re}\,g_2]$$
$$-\gamma[\langle\hat{S}\rangle-\beta\langle\hat{H}\rangle-\lambda_0-\lambda_1(\tfrac{1}{2}+\mathrm{e}^{-g_1})\mp2\,\mathrm{e}^{-g_1}\,\mathrm{Re}\,\lambda_2+\lambda_3 g_2\,\mathrm{e}^{-g_1}] \tag{89}$$

$\alpha=\pm\mathrm{i}$,

$$-\dot{g}_0-\tfrac{1}{2}\dot{g}_1+\dot{g}_1\,\mathrm{e}^{-g_1}(\pm2\,\mathrm{Im}\,g_2-1)\mp2\,\mathrm{e}^{-g_1}\,\mathrm{Im}\,\dot{g}_2$$
$$=\frac{2}{\hbar}[\mp h(t)(1-\mathrm{e}^{-g_1})-\mathrm{e}^{-g_1}(h(t)\,\mathrm{Im}\,g_2\mp\hbar\omega\,\mathrm{Re}\,g_2)]$$
$$-\gamma\beta[\hbar\omega(\tfrac{1}{2}+\mathrm{e}^{-g_1})-\mathrm{e}^{-g_1}(h\,\mathrm{Re}\,g_2\pm\hbar\omega\,\mathrm{Im}\,g_2)]$$
$$-\gamma[\langle\hat{S}\rangle-\beta\langle\hat{H}\rangle-\lambda_0-\lambda_1(\tfrac{1}{2}+\mathrm{e}^{-g_1})\mp2\mathrm{e}^{-g_1}\,\mathrm{Im}\,\lambda_2+\lambda_3 g_2\,\mathrm{e}^{-g_1}]. \tag{90}$$

Inserting (88) into (89) and adding the resulting expressions for $\alpha=\pm1$ leads to

$$\dot{g}_1=-\gamma g_1+\gamma\beta\hbar\omega \tag{91}$$

whereas subtraction gives

$$\mathrm{Re}\,\dot{g}_2=\omega\,\mathrm{Im}\,g_2+\frac{\beta\gamma h(t)}{2}(\mathrm{e}^{g_1}+1)-\gamma\,\mathrm{Re}\,g_2\left(\frac{g_1}{1-\mathrm{e}^{-g_1}}-\frac{\beta\hbar\omega}{2}\right). \tag{92}$$

Next we insert (88) into (90) and subtract the resultant equations for $\alpha=\pm\mathrm{i}$, yielding

$$\mathrm{Im}\,\dot{g}_2=-\omega\,\mathrm{Re}\,g_2-\frac{1}{\hbar}h(t)(1-\mathrm{e}^{g_1})-\gamma\,\mathrm{Im}\,g_2\left(\frac{g_1}{1-\mathrm{e}^{-g_1}}-\frac{\beta\hbar\omega}{2}\right). \tag{93}$$

For $g_2=\mathrm{Re}\,g_2+\mathrm{i}\,\mathrm{Im}\,g_2$ one gets with (92) and (93)

$$\dot{g}_2=-\mathrm{i}\omega g_2+h(t)\left(\frac{\beta\gamma}{2}(\mathrm{e}^{g_1}+1)+\frac{\mathrm{i}}{\hbar}(\mathrm{e}^{g_1}-1)\right)-\gamma g_2\left(\frac{g_1}{1-\mathrm{e}^{-g_1}}-\frac{\beta\hbar\omega}{2}\right)$$
$$=h(t)\left(\frac{\gamma\beta}{2}(\mathrm{e}^{g_1}+1)+\frac{\mathrm{i}}{\hbar}(\mathrm{e}^{g_1}-1)\right)-g_2\left[\gamma\left(\frac{g_1}{1-\mathrm{e}^{-g_1}}-\frac{\beta\hbar\omega}{2}\right)+\mathrm{i}\omega\right] \tag{94}$$

i.e. g_2 fulfills a linear inhomogeneous first-order differential equation. Equation (91) is solved by

$$g_1(t)=\beta\hbar\omega\left[1-\mathrm{e}^{-\gamma t}\left(1-\frac{g_1(0)}{\beta\hbar\omega}\right)\right]. \tag{95}$$

Using the abbreviations

$$X = h(t)\left(\frac{\gamma\beta}{2}(e^{g_1}+1)+\frac{i}{\hbar}(e^{g_1}-1)\right) = h(t)(a+ib) \tag{96}$$

$$Y = \gamma\left(\frac{g_1}{1-e^{-g_1}}-\frac{\beta\hbar\omega}{2}\right)+i\omega = c+i\omega \tag{97}$$

(94) reduces to

$$\dot{g}_2 = X - Yg_2 \tag{98}$$

with the formal solution

$$g_2(t) = \left(\int_0^t X(t')\exp\left(\int_0^{t'} Y(t'')\,dt''\right)dt'+C_0\right)\exp\left(-\int_0^t Y(t')\,dt'\right). \tag{99}$$

It should be noted that for $F(t)=0$, i.e. without a driving force, we get $X=0$ and the above equation gives an exponential decay of g_2. For the harmonic driving force (25), (99) again can be integrated in closed form, and Re g_2 and Im g_2 can then be determined by calculating the real and imaginary parts of (99). Let us first solve (99). Since $\dot{g}_1 = \gamma(\beta\hbar\omega - g_1)$ we have $g_1(t)\to\beta\hbar\omega$ for $t\to\infty$. Therefore, we put $g_1 = \text{constant} = \beta\hbar\omega$, i.e. we consider the oscillator at constant temperature or in the limit $t\to\infty$. Then we obtain

$$Y = \gamma\left(\frac{g_1}{1-e^{g_1}}-\frac{\beta\hbar\omega}{2}\right)+i\omega = \gamma\frac{\beta\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right)+i\omega \tag{100}$$

which converges to $\gamma+i\omega$ for $\beta\to 0$. With the abbreviation

$$f = \gamma\frac{\beta\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right) \tag{101}$$

one gets

$$\exp\left(-\int_0^t Y(t')\,dt'\right) = e^{-Yt} = e^{-ft-i\omega t}. \tag{102}$$

A short glance at (96) bearing $g_1 = \text{constant}$ in mind immediately gives

$$a = \frac{\gamma\beta}{2}(e^{g_1}+1) = \text{constant} \tag{103}$$

$$b = \frac{1}{\hbar}(e^{g_1}-1) = \text{constant} \tag{104}$$

and finally we get

$$g_2 = -\frac{F_0}{2}\sqrt{\frac{\hbar}{2m\omega}}(a+ib)\left(\int_0^t (e^{i\Omega t'}+e^{-i\Omega t'})\,e^{ft'+i\omega t'}\,dt'+C_0\right)e^{-ft-i\omega t}. \tag{105}$$

In the limit $t\to\infty$ the factor $C_0\,e^{-ft-i\omega t}$ can be omitted ($f>0$). But C_0 is determined by the initial value $g_2(0)$, so, at this certain point, information about the past of the system is lost. By direct integration we get

$$g_2 = -\frac{F_0}{2}\sqrt{\frac{\hbar}{2m\omega}}(a+ib)\left(\frac{e^{i\Omega t}-e^{-ft-i\omega t}}{i(\Omega+\omega)+f}+\frac{e^{-i\Omega t}-e^{-ft-i\omega t}}{i(\omega-\Omega)+f}\right) \tag{106}$$

and consequently considering again only the limit $t \to \infty$ of (106) we obtain

$$g_2 = -\frac{F_0}{2}\sqrt{\frac{\hbar}{2m\omega}}\,(a+\mathrm{i}b)\left(\frac{\mathrm{e}^{\mathrm{i}\Omega t}}{\mathrm{i}(\Omega+\omega)+f} + \frac{\mathrm{e}^{-\mathrm{i}\Omega t}}{\mathrm{i}(\omega-\Omega)+f}\right). \tag{107}$$

After lengthy but straightforward manipulations this can be rewritten as

$$g_2 = -F_0\sqrt{\frac{\hbar}{2m\omega}}\,\frac{b}{\omega}\,[B\cos(\Omega t-\phi) + \mathrm{i}\tilde{B}\cos(\Omega t-\psi)] \tag{108}$$

with

$$B = \frac{\{[(f^2+\omega^2)^2+\Omega^2(f^2-\omega^2)]^2 + f^2\Omega^2(\Omega^2+f^2+\omega^2)^2)\}^{1/2}}{(\Omega^2+f^2-\omega^2)^2+4f^2\omega^2} \tag{109}$$

$$\tilde{B} = \frac{\Omega\omega}{[(\Omega^2+f^2-\omega^2)^2+4f^2\omega^2]^{1/2}} \tag{110}$$

$$\tan\phi = \frac{f\Omega(\Omega^2+f^2+\omega^2)}{(f^2+\omega^2)^2+\Omega^2(f^2-\omega^2)} \tag{111}$$

$$\tan\psi = \frac{\Omega^2-f^2-\omega^2}{2f\Omega}. \tag{112}$$

This completes the algebraic solution of the evolution equation.

Let us now consider the behaviour of the expectation value $\langle\hat{x}\rangle$. With the representation $\hat{\rho} = \Pi_{i=0}^{3}\,\mathrm{e}^{-g_i\hat{\Gamma}_i}$ one gets (see appendix 1)

$$\mathrm{Tr}\,\hat{\rho}\hat{a} = -\frac{g_2}{\mathrm{e}^{g_1}-1} \tag{113}$$

$$\mathrm{Tr}\,\hat{\rho}\hat{a}^{\dagger} = -\frac{g_3\,\mathrm{e}^{g_1}}{\mathrm{e}^{g_1}-1} \tag{114}$$

and, because of $g_3\,\mathrm{e}^{g_1} = \overline{g_2}$, according to (73) this yields the following expression for $\langle\hat{x}\rangle$:

$$\langle\hat{x}\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\mathrm{Tr}\,[\hat{\rho}(\hat{a}^{\dagger}+\hat{a})] = -\sqrt{\frac{2\hbar}{m\omega}}\,\frac{\mathrm{Re}\,g_2}{\mathrm{e}^{g_1}-1}. \tag{115}$$

First we remark that without a driving force g_2 exponentially decreases as was already pointed out and so $\langle\hat{x}\rangle$ goes to zero. Inserting (104) and (108) we obtain an expression for the long-time limit $\langle\hat{x}\rangle_\infty(t)$:

$$\langle\hat{x}\rangle_\infty(t) = \frac{F_0B}{m\omega^2}\cos(\Omega t-\phi). \tag{116}$$

Notice that in the limit $f \to 0$ the amplitude $\hat{x}$ reduces to $[F_0/m(\Omega^2-\omega^2)]$ and the phase shift ϕ goes to zero, so that (116) becomes a solution of (53).

The amplitude $\hat{x} = F_0B/m\omega^2$ of $\langle\hat{x}\rangle_\infty(t)$ shows a typical resonance behaviour as a function of the driving frequency Ω, with a maximum close to

$$\Omega_{\mathrm{res}} \approx \sqrt{\omega^2-f^2} \tag{117}$$

where the value of $\hat{x}$ is given as

$$\hat{x}_{\text{res}} \approx \frac{F_0}{2m\omega^2}\sqrt{3+\left(\frac{\omega}{f}\right)^2}. \tag{118}$$

In the limit $\Omega \to 0$ the amplitude tends to the constant value of $F_0/m\omega^2$. As an illustrative example, figure 1 shows the amplitude B as a function of Ω/ω for four different values of the damping constant γ and fixed values of ω and $\beta = 1/kT$. We use $\omega = 10^{11}$ Hz and a temperature of 300 K, i.e. $\hbar\omega/kT \approx 3 \times 10^{-3}$, and $\gamma/\omega = 0.01$, 0.05, 0.1 and 0.2. Note that $f \approx \gamma$ in this high-temperature region. One observes the resonance behaviour discussed above. In figure 2 the corresponding phaseshifts ϕ are shown, which increase from zero for low driving frequencies, pass through $\pi/2$ at resonance and approach π for large driving frequencies.

From (115) a differential equation for the classical mean value $\langle\hat{x}\rangle$ can be derived in the same way as (53) was deduced. The algebraic manipulations are somewhat more

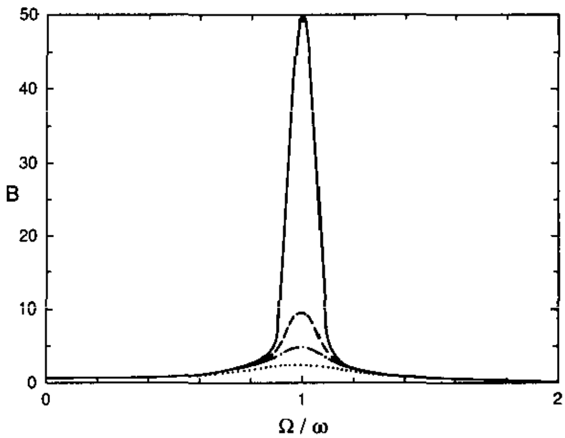

Figure 1. Dependence of the amplitude B on the driving frequency Ω for different damping constants: $f/\omega = 0.01$ (——), 0.05 (– – –), 0.1 (— · —), 0.2 (· · · ·).

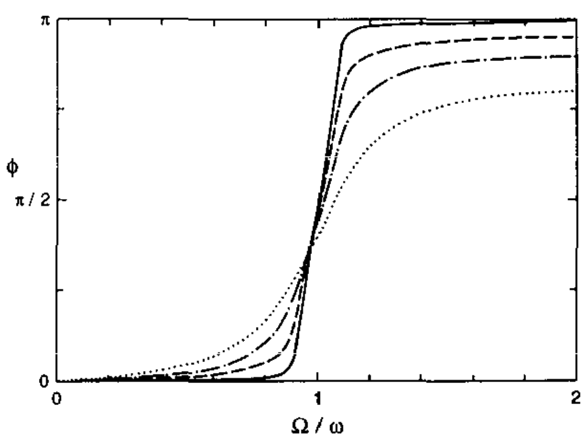

Figure 2. Dependence of the phaseshift ϕ on the driving frequency Ω for different damping constants (see figure 1).

involved so that we will briefly outline them. First we obtain

$$\frac{d^2}{dt^2}\langle\hat{x}\rangle = -\sqrt{\frac{2\hbar}{m\omega}}\,\frac{\mathrm{Re}\,\ddot{g}_2}{e^{\beta\hbar\omega}-1}$$

$$= -\sqrt{\frac{2\hbar}{m\omega}}\,\frac{\omega\,\mathrm{Im}\,\dot{g}_2 + a\dot{h} - \mathrm{Re}\,\dot{g}_2 f}{(e^{g_1}-1)}. \qquad (119)$$

Inserting (93) for $\mathrm{Im}\,\dot{g}_2$ and $h(t) = -\sqrt{\hbar/2m\omega}\,F_0 \cos\Omega t$ leads to

$$\frac{d^2}{dt^2}\langle\hat{x}\rangle = -\omega^2\langle\hat{x}\rangle - f\frac{d}{dt}\langle\hat{x}\rangle + \frac{F_0}{m}\cos\Omega t$$

$$+\sqrt{\frac{2\hbar}{m\omega}}\,\frac{1}{(e^{g_1}-1)}\left(\omega\,\mathrm{Im}\,g_2 f - a\Omega F_0\sqrt{\frac{\hbar}{2m\omega}}\sin\Omega t\right) \qquad (120)$$

and with $\mathrm{Im}\,g_2$ from (108) we obtain after some algebraic manipulations

$$\frac{d^2}{dt^2}\langle\hat{x}\rangle + f\frac{d}{dt}\langle\hat{x}\rangle + \omega^2\langle\hat{x}\rangle = \frac{F_0}{m}C\cos(\Omega t+\delta) \qquad (121)$$

with

$$C^2 = \left(1 - \frac{2f^2\Omega^2}{(\Omega^2+f^2-\omega^2)^2+4f^2\omega^2)}\right)^2 + f^2\Omega^2\left(\frac{\Omega^2-\omega^2-f^2}{(\Omega^2+f^2-\omega^2)^2+4f^2\omega^2}+\frac{1}{\omega^2}\right)^2 \qquad (122)$$

and

$$\tan\delta = \frac{f\Omega}{\omega^2}\,\frac{(\Omega^2+f^2)^2+\omega^2(f^2-\Omega^2-\omega^2)}{(\Omega^2-\omega^2)^2+f^2(2\omega^2+f^2)}. \qquad (123)$$

From this equation one can can see that the expectation value $\langle\hat{x}\rangle$ follows a classical evolution equation, where f defined in (101) plays the role of a classical damping constant.

A remarkable property of (121) is the fact that the damping is proportional to the 'velocity' $d\langle\hat{x}\rangle/dt$. A closer look at the derivation of (121) shows that this stems from the fact that in the term $[(\hat{S}-\langle\hat{S}\rangle)\hat{\rho} - \beta(\frac{1}{2}[\hat{H},\hat{\rho}]_+ - \langle\hat{H}\rangle\hat{\rho})]$ the entropy $\hat{S} = -\ln\hat{\rho}$ and also therefore the coefficient functions λ_i, $i = 1, 2, 3$, appear only linear.

Since we started with a force $F(t) = F_0\cos\Omega t$ acting on our particle, we would expect exactly the same force appearing in the differential equation for $\langle\hat{x}\rangle$, but only if we applied classical arguments and a classical damping process. Actually, the difference between this classical expectation and our result is a purely quantum mechanical effect related to the additional term in the von Neumann equation. The amplitude of the force appearing in (121) is modified by a quantum correction factor C and an additional phaseshift δ appears. The well known closed-form solution of the evolution equation (121) agrees with the result given in (116), in particular the relation

$$B = \frac{C\omega^2}{(\omega^2-\Omega^2)^2+(f\Omega)^2} \qquad (124)$$

is valid.

It is useful to rewrite the damping constant f as

$$f = \gamma\frac{U}{kT} \qquad (125)$$

where

$$U = \frac{\hbar\omega}{2} \coth\left(\frac{\hbar\omega}{2kT}\right) \tag{126}$$

is the equilibrium mean energy of the harmonic oscillator at temperature T. In the high-temperature limit as well as in the classical limit $\hbar \to 0$, the mean energy U approaches kT and f tends to the damping rate γ. For $T \to 0$ the damping term f diverges. This is a quantum phenomenon originating from the fact that U cannot fall below $\frac{1}{2}\hbar\omega$.

The amplitude C and the phaseshift δ of the force term appearing in (121) are illustrated in figures 3 and 4, calculated with the same parameters as for figures 1 and 2. The quantum correction C is different from unity only in a small region close to resonance, when the correction factor drops down to $\frac{1}{2}$. Here (and, of course, at low

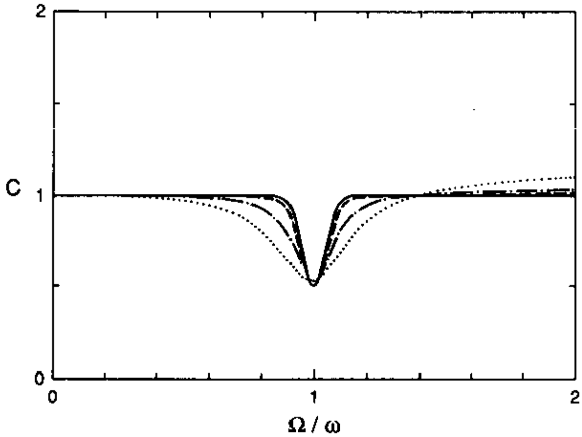

Figure 3. Quantum correction factor C of the force amplitude as a function of the driving frequency Ω for different damping constants (see figure 1).

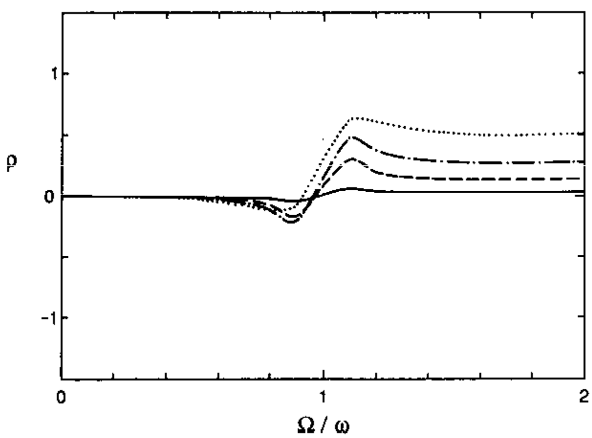

Figure 4. Quantum phaseshift δ of the force as a function of the driving frequency Ω for different damping constants (see figure 1).

temperatures) we expect differences between the classical and quantum behaviours. The phaseshift correction δ increaes with increasing γ.

In addition it will be instructive to write down the long-time behaviour of the expectation value of the energy. From (47) we find with help of the results (73)–(76) and (108)

$$\begin{aligned}\langle \hbar\omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})\rangle &= \hbar\omega\left(\frac{1}{2}+\frac{1}{e^{g_1}-1}+\frac{\lambda_2\lambda_3}{\lambda_1^2}\right)\\ &= U+\frac{\hbar\omega|g_2|^2}{(e^{g_1}-1)^2}\\ &= U+\frac{F_0^2}{2m\omega^2}[B^2\cos^2(\Omega t-\phi)+\tilde{B}^2\cos^2(\Omega t-\psi)] \qquad (127)\end{aligned}$$

i.e. the equilibrium value plus an oscillatory contribution from the force. Averaging over one period we obtain a contribution of $(F_0^2/4m\omega^2)$ $(B^2+\tilde{B}^2)$ from the force term. On resonance we have $B\approx\tilde{B}=\omega/2f$, $\phi=\pi/2$ and $\psi=0$, and therefore the average energy is

$$\langle \hbar\omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})\rangle = U+\frac{F_0^2}{8mf^2} \qquad (128)$$

which is constant in time.

Finally it should be noted that explicit expressions for the transition amplitudes $\langle n'|\hat{\rho}|n\rangle$ can be easily obtained from (67) using algebraic techniques developed in [39].

4. Concluding remarks

In this article we have studied the quantum mechanical time evolution of a linearly forced harmonic oscillator by means of a dissipative nonlinear von Neumann equation proposed in a preceding article (paper I). An analytic solution could be derived, which is quite remarkable because of the nonlinearity of the equations.

The present approach is entirely different from the major ones found in the literature [8–34], both in method and in the kind of results obtained. Usually, a wavefunction approach is considered, based on a canonical quantization procedure of some appropriate Hamiltonian which in turn is derived from a Lagrangian generating the classical equations of motion. But such a Lagrangian is not unique and so one has to deal with problems arising from these ambiguities. This problem is, however, successfully circumvented by the present approach, since it starts with the Hamiltonian of a classically undamped oscillator and the whole damping process is included by means of an additional term in the von Neuman equation. This term is derived from some simple consistency arguments (see paper I), without using further canonical quantization methods, which are clearly the source of all the problems mentioned above.

Even for the harmonic oscillator only in rare cases has the mean value $\langle\hat{x}\rangle$ of the position operator been worked out up to now. Svinin [9], Brinati and Mizrahi [17] and Remaud and Hernandez [18] derived a behaviour in accordance with the classical case of a damped harmonic oscillator, but without including any driving force. Such driving forces are considered by, for example, Khandekar and Lawande [13] and also in [20, 22, 25, 28], where various quantities such as the Feynman propagator or the transition amplitudes between oscillator eigenstates are calculated. These are of course also important and instructive but do not allow a direct comparison of the quantum behaviour with its classical counterpart in the simple way achieved in the present paper, where explicit expressions not only for the position $\langle\hat{x}\rangle$ but even for the phaseshift between driving force and resulting motion are derived, in surprising agreement with classical results.

But there are still a lot of open questions and further studies are clearly necessary, e.g. a more detailed discussion of the properties of the solution (different parameter ranges, classical and semiclassical limits, transition amplitudes, etc.) as well as a careful consideration of possible experimental tests of the predictions of the present model for dissipative time evolution. Work along these lines is in progress.

In conclusion it should be recalled that there are various mathematical and conceptual problems related to the derivation and interpretation of the generalized dissipative von Neumann equation (see the concluding remarks in paper I). It is hoped, however, that this equation will be a useful tool for a discription of quantum dissipation.

Acknowledgments

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Appendix 1. Calculation of the traces

In this appendix we will calculate Tr $\hat{\rho}$, Tr $\hat{a}^\dagger\hat{a}\hat{\rho}$, Tr $\hat{a}^\dagger\hat{\rho}$ and Tr $\hat{a}\hat{\rho}$ using the representation $\hat{\rho}=\Pi_{i=0}^3 \mathrm{e}^{-g_i\hat{\Gamma}_i}$ (Wei–Norman representation, see [39, 40]). We will use the complete orthonormal system of eigenfunctions $|n\rangle$ of $\hat{a}^\dagger\hat{a}$

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle \tag{A1.1}$$

with

$$\hat{a}^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle \tag{A1.2}$$

$$\hat{a}|n\rangle=\sqrt{n}|n-1\rangle \tag{A1.3}$$

and (e.g. see [38])

$$\langle m|\mathrm{e}^{g\hat{a}}|n\rangle=\begin{cases}\sqrt{\dfrac{n!}{m!}}\dfrac{g^{n-m}}{(n-m)!} & m\leqslant n\\ 0 & m>n\end{cases} \tag{A1.4}$$

$$\langle m|\mathrm{e}^{g\hat{a}^\dagger}|n\rangle=\begin{cases}\sqrt{\dfrac{m!}{n!}}\dfrac{g^{m-n}}{(m-n)!} & m\geqslant n\\ 0 & m<n.\end{cases} \tag{A1.5}$$

After calculating the traces the product of exponentials $\hat{\rho}=\pi_{i=0}^3 \mathrm{e}^{-g_i\hat{\Gamma}_i}$ is transformed into exponential form $\hat{\rho}=\exp(-\Sigma_{i=0}^3 \lambda_i\hat{\Gamma}_i)$ (Magnus representation, see [41]) and the coefficient functions are converted into each other.

First we evaluate $\mathrm{Tr}\,\hat{\rho}$:

$$\mathrm{Tr}\,\hat{\rho} = \mathrm{Tr}\,\mathrm{e}^{-g_I\hat{1}}\,\mathrm{e}^{-g_1(\hat{a}^\dagger\hat{a}+1/2)}\,\mathrm{e}^{-g_2\hat{a}^\dagger}\,\mathrm{e}^{-g_3\hat{a}}$$

$$= \sum_{n=0}^{\infty}\langle n|\mathrm{e}^{-g_0-g_1/2}\,\mathrm{e}^{-g_1\hat{a}^\dagger\hat{a}}\,\mathrm{e}^{-g_2\hat{a}^\dagger}\,\mathrm{e}^{-g_3\hat{a}}|n\rangle$$

$$= \mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}\mathrm{e}^{-g_1 n}\,\langle n|\mathrm{e}^{-g_2\hat{a}^\dagger}\,\mathrm{e}^{-g_3\hat{a}}|n\rangle$$

$$= \mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathrm{e}^{-g_1 n}\,\langle n|\mathrm{e}^{-g_2\hat{a}^\dagger}|m\rangle\langle m|\mathrm{e}^{-g_3\hat{a}}|n\rangle$$

$$= \mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}\mathrm{e}^{-g_1 n}\sum_{m=0}^{n}\frac{n!}{m!}\frac{(g_2g_3)^{n-m}}{[(n-m)!]^2}$$

$$= \mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}\mathrm{e}^{-g_1 n}\sum_{k=0}^{n}\frac{n!(g_2g_3)^k}{(n-k)!(k!)^2}. \tag{A1.6}$$

Using Pochhammer's symbol $(a)_b = a\cdot(a+1)\ldots(a+b-1)$ and introducing the Laguerre polynomials $L_n(x) = n!\sum_{k=0}^{n}(-n)_n/(k!)^2x^k$ and the abbreviations $y=\mathrm{e}^{-g_1}$ and $x=-g_2g_3$ this can be rewritten as

$$\mathrm{Tr}\,\hat{\rho} = \mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}y^n\sum_{k=0}^{n}\frac{(-n)_k}{(k!)^2}x^k \tag{A1.7}$$

$$= \mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}y^n\frac{L_n(x)}{n!}$$

$$= \mathrm{e}^{-g_0-g_1/2}\frac{1}{1-y}\mathrm{e}^{-xy/(1-y)}$$

$$= \frac{\mathrm{e}^{-g_0}}{2\sinh(g_1/2)}\exp\left(-\frac{g_2g_3}{1-\mathrm{e}^{-g_1}}\right). \tag{A1.8}$$

In a similar manner we calculate $\mathrm{Tr}\,\hat{\rho}\hat{a}^\dagger\hat{a}$:

$$\mathrm{Tr}\,\hat{\rho}\hat{a}^\dagger\hat{a} = \sum_{n=0}^{\infty}\langle n|\mathrm{e}^{-g_0-g_1/2}\,\mathrm{e}^{-g_1\hat{a}^\dagger\hat{a}}\,\mathrm{e}^{-g_2\hat{a}^\dagger}\,\mathrm{e}^{-g_3\hat{a}}\,\hat{a}^\dagger\hat{a}|n\rangle$$

$$= \mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}n\,\mathrm{e}^{-g_1 n}\langle n|\mathrm{e}^{-g_2\hat{a}^\dagger}\,\mathrm{e}^{-g_3\hat{a}}|n\rangle$$

$$= \mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}n\,\mathrm{e}^{-g_1 n}\frac{L_n(-g_2g_3)}{n!}$$

$$= \mathrm{e}^{-g_0-g_1/2}\,y\frac{\mathrm{d}}{\mathrm{d}y}\left(\sum_{n=0}^{\infty}y^n\frac{L_n(x)}{n!}\right)$$

$$= \mathrm{e}^{-g_0-g_1/2}\,y\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{1-y}\mathrm{e}^{-xy/(1-y)}\right)$$

$$= \mathrm{e}^{-g_0-g_1/2}\,y\left[\frac{1}{(1-y)^2}+\frac{1}{1-y}\left(-\frac{x}{1-y}-\frac{xy^2}{(1-y)^2}\right)\right]\mathrm{e}^{-xy/(1-y)}. \tag{A1.9}$$

After some simple algebra this leads to

$$\begin{aligned}\mathrm{Tr}\,\hat\rho\hat\Gamma_1&=\mathrm{Tr}\,\hat\rho(\hat a^\dagger\hat a+\tfrac12)\\&=\mathrm{e}^{-g_0}\exp\left(-\frac{g_2g_3}{1-\mathrm{e}^{g_1}}\right)\left(\frac{\cosh(g_1/2)}{4\sinh^2(g_1/2)}+\frac{g_2g_3}{8\sinh^3(g_1/2)}\right).\end{aligned}\tag{A1.10}$$

Finally, we evaluate $\mathrm{Tr}\,\hat\rho\hat\Gamma_2=\mathrm{Tr}\,\hat\rho\hat a^\dagger$:

$$\begin{aligned}\mathrm{Tr}\,\hat\rho\hat a^\dagger&=\sum_{n=0}^{\infty}\langle n|\mathrm{e}^{-g_0-g_1/2}\,\mathrm{e}^{-g_1\hat a^\dagger\hat a}\,\mathrm{e}^{-g_2\hat a^\dagger}\,\mathrm{e}^{-g_3\hat a}\,\hat a^\dagger|n\rangle\\
&=\mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}\langle n|\mathrm{e}^{-g_1\hat a^\dagger\hat a}\,\mathrm{e}^{-g_2\hat a^\dagger}\,\mathrm{e}^{-g_3\hat a}\sqrt{n+1}|n+1\rangle\\
&=\mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathrm{e}^{-g_1n}\sqrt{n+1}\langle n|\mathrm{e}^{-g_2\hat a^\dagger}|m\rangle\langle m|\mathrm{e}^{-g_3\hat a}|n+1\rangle\\
&=\mathrm{e}^{-g_0-g_1/2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathrm{e}^{-g_1n}\,(n+1)\,\frac{n!}{m!}\,\frac{(-g_2)^{n-m}(-g_3)^{n-m+1}}{(n-m)!(n-m+1)!}\\
&=\mathrm{e}^{-g_0-g_1/2}\,\mathrm{e}^{g_1}\sum_{j=1}^{\infty}\sum_{m=0}^{j-1}\mathrm{e}^{-g_1j}\,\frac{(-g_2)^{j-m-1}(-g_3)^{j-m}j!}{(j-m-1)!(j-m)!\,m!}\\
&=\mathrm{e}^{-g_0-g_1/2}\,\mathrm{e}^{g_1}\sum_{j=1}^{\infty}\mathrm{e}^{-g_1j}\sum_{k=1}^{j}\frac{(-g_2)^{k-1}(-g_3)^kj!}{(k-1)!(j-k)!\,k!}\\
&=g_3\,\mathrm{e}^{-g_0-g_1/2}\,\mathrm{e}^{g_1}\sum_{j=0}^{\infty}\mathrm{e}^{-g_1j}\sum_{k=1}^{j}\frac{k(-g_2g_3)^{k-1}(-j)_k}{(k!)^2}\\
&=g_3\,\mathrm{e}^{-g_0-g_1/2}\,\mathrm{e}^{g_1}\sum_{j=0}^{\infty}\mathrm{e}^{-g_1j}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{j}\frac{x^k(-j)_k}{(k!)^2}\right)\\
&=g_3\,\mathrm{e}^{-g_0-g_1/2}\,\mathrm{e}^{g_1}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{j=0}^{\infty}\mathrm{e}^{-g_1j}\,\frac{L_j(x)}{j!}\right)\\
&=g_3\,\mathrm{e}^{-g_0-g_1/2}\,\mathrm{e}^{g_1}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-y}\,\mathrm{e}^{-xy/(1-y)}\right)\\
&=-\frac{g_3\exp(-g_0+\frac12g_1)}{4\sinh^2(g_1/2)}\exp\left(-\frac{g_2g_3}{1-\mathrm{e}^{g_1}}\right).\end{aligned}\tag{A1.11}$$

This implies

$$\mathrm{Tr}\,\hat\rho\hat\Gamma_3=\mathrm{Tr}\,\hat\rho\hat a=-\frac{g_2\exp(-g_0-\frac12g_1)}{4\sinh^2(g_1/2)}\exp\left(-\frac{g_2g_3}{1-\mathrm{e}^{g_1}}\right)\tag{A1.12}$$

Furthermore the requirement $\mathrm{Tr}\,\hat\rho=1$ gives the constraint

$$\mathrm{e}^{-g_0}\exp\left(-\frac{g_2g_3}{1-\mathrm{e}^{g_1}}\right)=2\sinh\left(\frac{g_1}{2}\right)\tag{A1.13}$$

and the trace formulae can be simplified to

$$\mathrm{Tr}\,\hat\rho\hat\Gamma_1=\tfrac12+\frac{1}{\mathrm{e}^{g_1}-1}+\frac{g_2g_3}{4\sinh^2(g_1/2)}\tag{A1.14}$$

$$\mathrm{Tr}\,\hat\rho\hat\Gamma_2=-\frac{g_3\,\mathrm{e}^{g_1}}{\mathrm{e}^{g_1}-1}\tag{A1.15}$$

$$\mathrm{Tr}\,\hat\rho\hat\Gamma_3=-\frac{g_2}{\mathrm{e}^{g_1}-1}.\tag{A1.16}$$

For converting the coefficient functions g_i of the representation $\hat{\rho}=\Pi_{i=0}^{3} e^{-g_i\hat{\Gamma}_i}$ into the coefficient functions λ_i of the representation $\hat{\rho}=\exp(-\Sigma_{i=0}^{3}\lambda_i\hat{\Gamma}_i)$ we first make a similarity transformation $\hat{\rho}^{-1}\hat{\Gamma}_i\hat{\rho}$ in both representations. A comparison of the results yields the desired expressions to convert the coefficient functions.

Following Wilcox [35] ((6.28) and (6.26)) one has $(\Sigma=\Sigma_{i=0}^{3}\lambda_i\hat{\Gamma}_i)$

$$e^{\Sigma}\hat{\Gamma}_1 e^{-\Sigma}=\hat{\Gamma}_1+\left(\frac{\lambda_3}{\lambda_1}(1-e^{-\lambda_1})\right)\hat{\Gamma}_3+\left(\frac{\lambda_2}{\lambda_1}(1-e^{\lambda_1})\right)\hat{\Gamma}_2-\frac{\lambda_2\lambda_3}{\lambda_1^2}4\sinh^2\left(\frac{\lambda_1}{2}\right)\hat{1} \tag{A1.17, A1.18}$$

$$e^{\Sigma}\hat{\Gamma}_2 e^{-\Sigma}=e^{\lambda_1}\hat{\Gamma}_2+\frac{\lambda_3}{\lambda_1}(e^{\lambda_1}-1)\hat{1} \tag{A1.19}$$

$$e^{\Sigma}\hat{\Gamma}_3 e^{-\Sigma}=e^{-\lambda_1}\hat{\Gamma}_3+\frac{\lambda_2}{\lambda_1}(e^{-\lambda_1}-1)\hat{1}. \tag{A1.20}$$

According to [38] (see table II), we have $(\Pi=\Pi_{i=0}^{3} e^{-g_i\hat{\Gamma}_i})$

$$\Pi^{-1}\hat{\Gamma}_1\Pi=\hat{\Gamma}_1+g_3\hat{\Gamma}_3-g_2\hat{\Gamma}_2-g_2g_3\hat{1} \tag{A1.21}$$

$$\Pi^{-1}\hat{\Gamma}_2\Pi=e^{g_1}\hat{\Gamma}_2+g_3 e^{g_1}\hat{1} \tag{A1.22}$$

$$\Pi^{-1}\hat{\Gamma}_3\Pi=e^{-g_1}\hat{\Gamma}_3-g_2 e^{-g_1}\hat{1}. \tag{A1.23}$$

Since the $\hat{\rho}^{-1}\hat{\Gamma}_i\hat{\rho}$ do not depend on the representation of $\hat{\rho}$ a comparison of the coefficients gives

$$g_1=\lambda_1 \tag{A1.24}$$

$$g_2=\frac{\lambda_2}{\lambda_1}(e^{\lambda_1}-1) \tag{A1.25}$$

$$g_3=\frac{\lambda_3}{\lambda_1}(1-e^{-\lambda_1}) \tag{A1.26}$$

and therefore the surprisingly simple results

$$\mathrm{Tr}\,\hat{\rho}=1 \tag{A1.27}$$

$$\mathrm{Tr}\,\hat{\rho}\hat{\Gamma}_1=\frac{1}{2}+\frac{1}{e^{\lambda_1}-1}+\frac{\lambda_2\lambda_3}{\lambda_1^2} \tag{A1.28}$$

$$\mathrm{Tr}\,\hat{\rho}\hat{\Gamma}_2=-\frac{\lambda_3}{\lambda_1} \tag{A1.29}$$

$$\mathrm{Tr}\,\hat{\rho}\hat{\Gamma}_3=-\frac{\lambda_2}{\lambda_1}. \tag{A1.30}$$

The method used above is not applicable to convert g_0 into λ_0. Using simple group theoretic methods Gilmore [42] has deduced an expression for $\mathrm{Tr}[\exp(-\Sigma_{i=0}^{3}\lambda_i\hat{\Gamma}_i)]$:

$$\mathrm{Tr}\left[\exp\left(-\sum_{i=0}^{3}\lambda_i\hat{\Gamma}_i\right)\right]=\frac{\exp(\lambda_2\lambda_3/\lambda_1-\lambda_0)}{2\sinh(\lambda_1/2)} \tag{A1.31}$$

and therefore

$$g_0=\lambda_0+\frac{\lambda_2\lambda_3}{\lambda_1^2}(1-\lambda_1-e^{-\lambda_1}). \tag{A1.32}$$

Appendix 2. Calculation of special normal-order forms

In order to omit the matrix element $\langle\alpha|\hat{\rho}|\alpha\rangle$ from (79) we have to convert $\hat{\rho}\hat{a}^\dagger$ into $\hat{a}^\dagger\hat{\rho}$, $\hat{a}\hat{\rho}$ into $\hat{\rho}\hat{a}$ and $e^{-g_0-g_1/2}\,e^{-g_1\hat{a}^\dagger\hat{a}}\,\hat{a}^\dagger\,e^{-g_2\hat{a}^\dagger}\,e^{-g_3\hat{a}}$ into $\hat{a}^\dagger\hat{\rho}$ using normal ordering techniques as described by Louisell [43]:

$$\begin{aligned}\hat{a}\hat{\rho} &= \hat{a}\,e^{-g_0\hat{1}}\,e^{-g_1(\hat{a}^\dagger\hat{a}+1/2)}\,e^{-g_2\hat{a}^\dagger}\,e^{-g_3\hat{a}}\\ &= e^{-g_0-g_1/2}\,\hat{a}\,e^{-g_1\hat{a}^\dagger\hat{a}}\,e^{-g_2\hat{a}^\dagger}\,e^{-g_3\hat{a}}\\ &= e^{-g_0-g_1/2}\,e^{-g_1\hat{a}^\dagger\hat{a}}\hat{a}\,e^{-g_1}\,e^{-g_2\hat{a}^\dagger}\,e^{-g_3\hat{a}}\\ &= e^{-g_0-g_1/2}\,e^{-g_1\hat{a}^\dagger\hat{a}}\,e^{-g_2\hat{a}^\dagger}\,(\hat{a}-g_2)\,e^{-g_1}\,e^{-g_3\hat{a}}\\ &= \hat{\rho}(\hat{a}-g_2)\,e^{-g_1}\end{aligned}\tag{A2.1}$$

which implies

$$\langle\alpha|\hat{a}\hat{\rho}|\alpha\rangle=\langle\alpha|\hat{\rho}|\alpha\rangle(\alpha-g_2)\,e^{-g_1}.\tag{A2.2}$$

From $(\hat{a}\hat{\rho})^\dagger=\hat{\rho}\hat{a}^\dagger$ one immediately deduces

$$\langle\alpha|\hat{\rho}\hat{a}^\dagger|\alpha\rangle=\langle\alpha|\hat{\rho}|\alpha\rangle(\bar{\alpha}-\overline{g_2})\,e^{-g_1}\tag{A2.3}$$

and

$$e^{-g_0-g_1/2}\,e^{-g_1\hat{a}^\dagger\hat{a}}\hat{a}^\dagger\,e^{-g_2\hat{a}^\dagger}\,e^{-g_3\hat{a}}=e^{-g_0-g_1/2}\,\hat{a}^\dagger\,e^{-g_1}\,e^{-g_1\hat{a}^\dagger\hat{a}}\,e^{-g_2\hat{a}^\dagger}\,e^{-g_3\hat{a}}=\hat{a}^\dagger\,e^{-g_1}\,\hat{\rho}\tag{A2.4}$$

finally yields

$$\langle\alpha|e^{-g_0-g_1/2}\,e^{-g_1\hat{a}^\dagger\hat{a}}\,\hat{a}^\dagger\,e^{-g_2\hat{a}^\dagger}\,e^{-g_3\hat{a}}\,|\alpha\rangle=\langle\alpha|\hat{a}^\dagger\,e^{-g_1}\,\hat{\rho}|\alpha\rangle=\bar{\alpha}\,e^{-g_1}\,\langle\alpha|\hat{\rho}|\alpha\rangle.\tag{A2.5}$$

In addition one obtains

$$\begin{aligned}\langle\alpha|\hat{\rho}|\alpha\rangle &= e^{-g_0-g_1/2}\langle\alpha|e^{-g_1\hat{a}^\dagger\hat{a}}\,e^{-g_2\hat{a}^\dagger}\,e^{-g_3\hat{a}}|\alpha\rangle\\ &= e^{-g_0-g_1/2}\,e^{-g_3\alpha}\langle\alpha|e^{-g_1\hat{a}^\dagger\hat{a}}\,e^{-g_2\hat{a}^\dagger}\,|\alpha\rangle\\ &= e^{-g_0-g_1/2}\,e^{-g_3\alpha}\langle\alpha|e^{-g_2\hat{a}^\dagger\exp(-g_1)}\,e^{-g_1\hat{a}^\dagger\hat{a}}\,|\alpha\rangle\\ &= e^{-g_0-g_1/2}\,e^{-g_3\alpha}\langle\alpha|e^{-g_2\hat{a}^\dagger\exp(-g_1)}\sum_{l=0}^{\infty}\frac{(e^{-g_1}-1)^l}{l!}(\hat{a}^\dagger)^l\hat{a}^l|\alpha\rangle\\ &= e^{-g_0-g_1/2}\,e^{-g_3\alpha}\,e^{-g_2\bar{\alpha}\exp(-g_1)}\sum_{l=0}^{\infty}\frac{(e^{-g_1})-1)^l}{l!}(\bar{\alpha})^l\alpha^l\\ &= e[-g_0-\tfrac{1}{2}g_1-g_3\alpha-g_2\bar{\alpha}\,e^{-g_1}+(e^{-g_1}-1)\bar{\alpha}\alpha]\\ &\neq 0 \qquad \forall\alpha.\end{aligned}\tag{A2.6}$$

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